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Full Length Article

Generalized fuzzy *b*-closed and generalized ⋆-fuzzy *b*-closed sets in double fuzzy topological spaces

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ABSTRACT

The purpose of this paper is to introduce and study a new class of fuzzy sets called (r, s)-generalized fuzzy *b*-closed sets and (r, s)-generalized ⋆-fuzzy *b*-closed sets in double fuzzy topological spaces. Furthermore, the relationships between the new concepts are introduced and established with some interesting examples.

1. Introduction

A progressive development of fuzzy sets [1] has been made to discover the fuzzy analogues of the crisp sets theory. On the other hand, the idea of intuitionistic fuzzy sets was first introduced by Atanassov [2]. Later on, Çoker [3] presented the notion of intuitionistic fuzzy topology. Samanta and Mondal [4], introduced and characterized the intuitionistic gradation of openness of fuzzy sets which is a generalization of smooth topology and the topology of intuitionistic fuzzy sets. The name "intuitionistic" is discontinued in mathematics and applications. Garcia and Rodabaugh [5] concluded that they work under the name "double".

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In 2009, Omari and Noorani [6] introduced generalized b-closed sets (briefly, gb-closed) in general topology. As a generalization of the results in References 6 and 7, we introduce and study (r, s)-generalized fuzzy b-closed sets in double fuzzy topological spaces, then a new class of fuzzy sets between an (r, s)-fuzzy b-closed sets and an (r, s)-generalized fuzzy b-closed sets namely (r, s)-generalized $\star$-fuzzy b-closed sets is introduced and investigated. Finally, the relationships between (r, s)-generalized fuzzy b-closed and (r, s)-generalized $\star$-fuzzy b-closed sets are introduced and established with some interesting counter examples.

2. Preliminaries

Throughout this paper, X will be a non-empty set, $I = [0, 1]$, $I_0 = (0, 1]$ and $I_1 = [0, 1)$. A fuzzy set λ is quasi-coincident with a fuzzy set μ (denoted by, $\lambda q \mu$) iff there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$ and they are not quasi-coincident otherwise (denoted by, $\lambda \bar{q} \mu$). The family of all fuzzy sets on X is denoted by I^X. By $\underline{0}$ and $\underline{1}$, we denote the smallest and the greatest fuzzy sets on X. For a fuzzy set $\lambda \in I^X$, $\underline{1} - \lambda$ denotes its complement. All other notations are standard notations of fuzzy set theory.

Now, we recall the following definitions which are useful in the sequel.

Definition 2.1. (see [4]) A double fuzzy topology (τ, τ^*) on X is a pair of maps $\tau, \tau^* : I^X \to I$, which satisfies the following properties:

(O1) $\tau(\lambda) \le \underline{1} - \tau^*(\lambda)$ for each $\lambda \in I^X$.

(O2) $\tau(\lambda_1 \wedge \lambda_2) \ge \tau(\lambda_1) \wedge \tau(\lambda_2)$ and $\tau^*(\lambda_1 \wedge \lambda_2) \le \tau^*(\lambda_1) \vee \tau^*(\lambda_2)$ for each $\lambda_1, \lambda_2 \in I^X$.

(O3) $\tau(\vee_{i\in\Gamma} \lambda_i) \ge \wedge_{i\in\Gamma} \tau(\lambda_i)$ and $\tau^*(\vee_{i\in\Gamma} \lambda_i) \le \vee_{i\in\Gamma} \tau^*(\lambda_i)$ for each $\lambda_i \in I^X, i \in \Gamma$.

The triplet (X, τ, τ^*) is called a double fuzzy topological space (briefly, dfts). A fuzzy set λ is called an (r, s)-fuzzy open (briefly, (r, s)-fo) if $\tau(\lambda) \ge r$ and $\tau^*(\lambda) \le s$. A fuzzy set λ is called an (r, s)-fuzzy closed (briefly, (r, s)-fc) set iff $\underline{1} - \lambda$ is an (r, s)-fo set.

Theorem 2.1. (see [8]) Let (X, τ, τ^*) be a dfts. Then double fuzzy closure operator and double fuzzy interior operator of $\lambda \in I^X$ are defined by

$$C_{\tau,\tau^*}(\lambda, r, s) = \wedge\{\mu \in I^X \mid \lambda \le \mu, \tau(\underline{1} - \mu) \ge r, \tau^*(\underline{1} - \mu) \le s\},$$

$$I_{\tau,\tau^*}(\lambda, r, s) = \vee\{\mu \in I^X \mid \mu \le \lambda, \tau(\mu) \ge r, \tau^*(\mu) \le s\}.$$

Where $r \in I_0$ and $s \in I_1$ such that $r + s \le 1$.

Definition 2.2. Let (X, τ, τ^*) be a dfts. For each $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$. A fuzzy set λ is called:

1. An (r, s)-fuzzy semiopen (see [9]) (briefly, (r, s)-fso) if $\lambda \le C_{\tau,\tau^*}(I_{\tau,\tau^*}(\lambda, r, s), r, s)$. λ is called an (r, s)-fuzzy semi closed (briefly, (r, s)-fsc) iff $\underline{1} - \lambda$ is an (r, s)-fso set.
2. An (r, s)-generalized fuzzy closed (see [10]) (briefly, (r, s)-gfc) if $C_{\tau,\tau^*}(\lambda, r, s) \le \mu$, $\lambda \le \mu$, $\tau(\mu) \ge r$ and $\tau^*(\mu) \le s$. λ is called an (r, s)-generalized fuzzy open (briefly, (r, s)-gfo) iff $\underline{1} - \lambda$ is (r, s)-gfc set.

Definition 2.3. (see [11,12]) Let (X, τ, τ^*) be a dfts. For each $\lambda, \mu \in I^X$ and $r \in I_0$, $s \in I_1$. Then, a fuzzy set λ is said to be (r, s)-fuzzy generalized $\psi\rho$-closed (briefly, (r, s)-fg$\psi\rho$-closed) if $\psi C_{\tau,\tau^*}(\lambda, r, s) \le \mu$ such that $\lambda \le \mu$ and μ is (r, s)-fuzzy ρ-open set. λ is called (r, s)-fuzzy generalized $\psi\rho$-open (briefly, (r, s)-fg$\psi\rho$-open) iff $\underline{1} - \lambda$ is (r, s)-fg$\psi\rho$-closed set.

3. (r, s)-generalized fuzzy b-closed sets

In this section, we introduce and study some basic properties of a new class of fuzzy sets called an (r, s)-fuzzy b-closed sets and an (r, s)-generalized fuzzy b-closed.

Definition 3.1. Let (X, τ, τ^*) be a dfts. For each $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$. A fuzzy set λ is called:

1. An (r, s)-fuzzy b-closed (briefly, (r, s)-fbc) if

$$\lambda \ge (I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, r, s), r, s)) \wedge (C_{\tau,\tau^*}(I_{\tau,\tau^*}(\lambda, r, s), r, s)).$$

λ is called an (r, s)-fuzzy b-open (briefly, (r, s)-fbo) iff $\underline{1} - \lambda$ is (r, s)-fbc set.

2. An (r, s)-generalized fuzzy b-closed (briefly, (r, s)-gfbc) if $bC_{\tau,\tau^*}(\lambda, r, s) \le \mu$, $\lambda \le \mu$, $\tau(\mu) \ge r$ and $\tau^*(\mu) \le s$. λ is called an (r, s)-generalized fuzzy b-open (briefly, (r, s)-gfbo) iff $\underline{1} - \lambda$ is (r, s)-gfbc set.

Definition 3.2. Let (X, τ, τ^*) be a dfts. Then double fuzzy b-closure operator and double fuzzy b-interior operator of $\lambda \in I^X$ are defined by

$$bC_{\tau,\tau^*}(\lambda, r, s) = \wedge\{\mu \in I^X \mid \lambda \le \mu \text{ and } \mu \text{ is } (r, s)\text{-fbc}\},$$

$$bI_{\tau,\tau^*}(\lambda, r, s) = \vee\{\mu \in I^X \mid \mu \le \lambda \text{ and } \mu \text{ is } (r, s)\text{-fbo}\}.$$

Where $r \in I_0$ and $s \in I_1$ such that $r + s \le 1$.

Remark 3.1. Every (r, s)-fbc set is an (r, s)-gfbc set.

The converse of the above remark may be not true as shown by the following example.

Example 3.1. Let $X = \{a, b\}$. Defined μ, α and β by:

$$\mu(a) = 0.3, \quad \mu(b) = 0.4,$$

$$\alpha(a) = 0.4, \quad \alpha(b) = 0.5,$$

$$\beta(a) = 0.3, \quad \beta(b) = 0.7,$$

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases} \qquad \tau^*(\lambda) = \begin{cases} 0, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ 1, & \text{otherwise.} \end{cases}$$

Then β is an $(\frac{1}{2},\frac{1}{2})$-gfbc set but not an $(\frac{1}{2},\frac{1}{2})$-fbc set.

Definition 3.3. Let (X, τ, τ^*) be a dfts, $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$. λ is called an (r, s)-fuzzy b-Q-neighborhood of $x_t \in P_t(X)$ if there exists an (r, s)-fbo set $\mu \in I^X$ such that $x_t q\mu$ and $\mu \le \lambda$.

The family of all (r, s)-fuzzy b-Q-neighborhood of x_t denoted by b-$Q(x_t, r, s)$.

Theorem 3.1. Let (X, τ, τ^*) be a dfts. Then for each $\lambda, \mu \in I^X$, $r \in I_0$ and $s \in I_1$, the operator bC_{τ,τ^*} satisfies the following statements:

(C1)	$bC_{\tau,\tau^*}(\underline{0}, r, s) = \underline{0}$, $bC_{\tau,\tau^*}(\underline{1}, r, s) = \underline{1}$,
(C2)	$\lambda \le bC_{\tau,\tau^*}(\lambda, r, s)$,
(C3)	If $\lambda \le \mu$, then $bC_{\tau,\tau^*}(\lambda, r, s) \le bC_{\tau,\tau^*}(\mu, r, s)$,
(C4)	If λ is an (r, s)-fbc, then $\lambda = bC_{\tau,\tau^*}(\lambda, r, s)$,
(C5)	If μ is an (r, s)-fbo, then $\mu q\lambda$ iff $\mu q bC_{\tau,\tau^*}(\lambda, r, s)$,
(C6)	$bC_{\tau,\tau^*}(bC_{\tau,\tau^*}(\lambda, r, s), r, s) = bC_{\tau,\tau^*}(\lambda, r, s)$,
(C7)	$bC_{\tau,\tau^*}(\lambda, r, s) \vee bC_{\tau,\tau^*}(\mu, r, s) \le bC_{\tau,\tau^*}(\lambda \vee \mu, r, s)$,
(C8)	$bC_{\tau,\tau^*}(\lambda, r, s) \wedge bC_{\tau,\tau^*}(\mu, r, s) \ge bC_{\tau,\tau^*}(\lambda \wedge \mu, r, s)$,

Proof. (1), (2), (3), and (4) are proved easily.

(5) Let $\mu\bar{q}\lambda$ and μ is an (r, s)-fbo set, then $\lambda \le \underline{1} - \mu$. But we have, $\mu q\lambda$ iff $\mu q bC_{\tau,\tau^*}(\lambda, r, s)$ and

$$bC_{\tau,\tau^*}(\lambda, r, s) \le bC_{\tau,\tau^*}(\underline{1} - \mu, r, s) = \underline{1} - \mu,$$

so $\mu\bar{q}bC_{\tau,\tau^*}(\lambda, r, s)$, which is contradiction. Then $\mu q\lambda$ iff $\mu q bC_{\tau,\tau^*}(\lambda, r, s)$.

(6) Let x_t be a fuzzy point such that $x_t \not\le bC_{\tau,\tau^*}(\lambda, r, s)$. Then there is an (r, s)-fuzzy b-Q neighborhood μ of x_t such that $\mu\bar{q}\lambda$. But by (5), we have an (r, s)-fuzzy b-Q-neighborhood μ of x_t such that

$$\mu\bar{q}bC_{\tau,\tau^*}(\lambda, r, s)$$

Also,

$$x_t \not\le bC_{\tau,\tau^*}(bC_{\tau,\tau^*}(\lambda, r, s), r, s).$$

Then

$$bC_{\tau,\tau^*}(bC_{\tau,\tau^*}(\lambda, r, s), r, s) \le bC_{\tau,\tau^*}(\lambda, r, s).$$

But we have,

$$bC_{\tau,\tau^*}(bC_{\tau,\tau^*}(\lambda, r, s), r, s) \ge bC_{\tau,\tau^*}(\lambda, r, s).$$

Therefore

$$bC_{\tau,\tau^*}(bC_{\tau,\tau^*}(\lambda, r, s), r, s) = bC_{\tau,\tau^*}(\lambda, r, s).$$

(7) and (8) are obvious.

Theorem 3.2. Let (X, τ, τ^*) be a dfts. Then for each $\lambda, \mu \in I^X$, $r \in I_0$ and $s \in I_1$, the operator bI_{τ,τ^*} satisfies the following statements:

1. $bI_{\tau,\tau^*}(\underline{1} - \lambda, r, s) = \underline{1} - bC_{\tau,\tau^*}(\lambda, r, s)$, $bC_{\tau,\tau^*}(\underline{1} - \lambda, r, s) = \underline{1} - bI_{\tau,\tau^*}(\lambda, r, s)$,
2. $bI_{\tau,\tau^*}(\underline{0}, r, s) = \underline{0}$, $bI_{\tau,\tau^*}(\underline{1}, r, s) = \underline{1}$,
3. $bI_{\tau,\tau^*}(\lambda, r, s) \le \lambda$,
4. If λ is an (r, s)-fbo, then $\lambda = bI_{\tau,\tau^*}(\lambda, r, s)$,
5. If $\lambda \le \mu$, then $bI_{\tau,\tau^*}(\lambda, r, s) \le bI_{\tau,\tau^*}(\mu, r, s)$,
6. $bI_{\tau,\tau^*}(bI_{\tau,\tau^*}(\lambda, r, s), r, s) = bI_{\tau,\tau^*}(\lambda, r, s)$,
7. $bI_{\tau,\tau^*}(\lambda \vee \mu, r, s) \ge bI_{\tau,\tau^*}(\lambda, r, s) \vee bI_{\tau,\tau^*}(\mu, r, s)$,
8. $bI_{\tau,\tau^*}(\lambda \wedge \mu, r, s) \le bI_{\tau,\tau^*}(\lambda, r, s) \wedge bI_{\tau,\tau^*}(\mu, r, s)$.

Proof. It is similar to Theorem 3.1.

Theorem 3.3. Let (X, τ, τ^*) be a dfts. $\lambda \in I^X$ is (r, s)-gfbo set, $r \in I_0$ and $s \in I_1$ if and only if $\mu \le bI_{\tau,\tau^*}(\lambda, r, s)$ whenever $\mu \le \lambda$, $\tau(\underline{1} - \mu) \ge r$ and $\tau^*(\underline{1} - \mu) \le s$.

Proof. Suppose that λ is an (r, s)-gfbo set in I^X, and let $\tau(\underline{1} - \mu) \ge r$ and $\tau^*(\underline{1} - \mu) \le s$ such that $\mu \le \lambda$. By the definition, $\underline{1} - \lambda$ is an (r, s)-gfbc set in I^X. So,

$$bC_{\tau,\tau^*}(\underline{1} - \lambda, r, s) \le \underline{1} - \mu$$

Also,

$$\underline{1} - bI_{\tau,\tau^*}(\lambda, r, s) \le \underline{1} - \mu.$$

And then,

$$\mu \le bI_{\tau,\tau^*}(\lambda, r, s).$$

Conversely, let $\mu \le \lambda$, $\tau(\underline{1} - \mu) \ge r$ and $\tau^*(\underline{1} - \mu) \le s$, $r \in I_0$ and $s \in I_1$ such that $\mu \le bI_{\tau,\tau^*}(\lambda, r, s)$. Now

$$\underline{1} - bI_{\tau,\tau^*}(\lambda, r, s) \le \underline{1} - \mu,$$

Thus

$$bC_{\tau,\tau^*}(\underline{1} - \lambda, r, s) \le \underline{1} - \mu.$$

That is, $\underline{1} - \lambda$ is an (r, s)-gfbc set, then λ is an (r, s)-gfbo set.

Theorem 3.4. Let (X, τ, τ^*) be a dfts, $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$. If λ is an (r, s)-gfbc set, then

1. $bC_{\tau,\tau^*}(\lambda, r, s) - \lambda$ does not contain any non-zero (r, s)-fc sets.
2. λ is an (r, s)-fbc iff $bC_{\tau,\tau^*}(\lambda, r, s) - \lambda$ is (r, s)-fc.
3. μ is (r, s)-gfbc set for each set $\mu \in I^X$ such that $\lambda \le \mu \le bC_{\tau,\tau^*}(\lambda, r, s)$.
4. For each (r, s)-fo set $\mu \in I^X$ such that $\mu \le \lambda$, μ is an (r, s)-gfbc relative to λ if and only if μ is an (r, s)-gfbc in I^X.
5. For each an (r, s)-fbo set $\mu \in I^X$ such that $bC_{\tau,\tau^*}(\lambda, r, s)\bar{q}\mu$ iff $\lambda\bar{q}\mu$.

Proof. (1) Suppose that $\tau(\underline{1} - \mu) \ge r$ and $\tau^*(\underline{1} - \mu) \le s$, $r \in I_0$ and $s \in I_1$ such that $\mu \le bC_{\tau,\tau^*}(\lambda, r, s) - \lambda$ whenever $\lambda \in I^X$ is an (r, s)-gfbc set. Since $\underline{1} - \mu$ is an (r, s)-fo set,

$$\lambda \le (\underline{1}-\mu) \Rightarrow bC_{\tau,\tau^*}(\lambda,r,s) \le (\underline{1}-\mu)$$
$$\Rightarrow \mu \le (\underline{1} - bC_{\tau,\tau^*}(\lambda,r,s))$$
$$\Rightarrow \mu \le (\underline{1} - bC_{\tau,\tau^*}(\lambda,r,s)) \wedge (bC_{\tau,\tau^*}(\lambda,r,s)-\lambda)$$
$$= \underline{0}$$

and hence $\mu = \underline{0}$ which is a contradiction. Then $bC_{\tau,\tau^*}(\lambda,r,s)-\lambda$ does not contain any non-zero (r, s)-fc sets.

(2) Let λ be an (r, s)-gfbc set. So, for each $r \in I_0$ and $s \in I_1$ if λ is an (r, s)-fbc set then,

$$bC_{\tau,\tau^*}(\lambda,r,s)-\lambda = \underline{0}$$

which is an (r, s)-fc set.

Conversely, suppose that $bC_{\tau,\tau^*}(\lambda,r,s)-\lambda$ is an (r, s)-fc set. Then by (1), $bC_{\tau,\tau^*}(\lambda,r,s)-\lambda$ does not contain any non-zero an (r, s)-fc set. But $bC_{\tau,\tau^*}(\lambda,r,s)-\lambda$ is an (r, s)-fc set, then

$$bC_{\tau,\tau^*}(\lambda,r,s)-\lambda = \underline{0} \Rightarrow \lambda = bC_{\tau,\tau^*}(\lambda,r,s).$$

So, λ is an (r, s)-fbc set.

(3) Suppose that $\tau(\alpha) \ge r$ and $\tau^*(\alpha) \le s$ where $r \in I_0$ and $s \in I_1$ such that $\mu \le \alpha$ and let λ be an (r, s)-gfbc set such that $\lambda \le \alpha$. Then

$$bC_{\tau,\tau^*}(\lambda,r,s) \le \alpha.$$

So,

$$bC_{\tau,\tau^*}(\lambda,r,s) = bC_{\tau,\tau^*}(\mu,r,s),$$

Therefore

$$bC_{\tau,\tau^*}(\mu,r,s) \le \alpha.$$

So, μ is an (r, s)-gfbc set.

(4) Let λ be an (r, s)-gfbc and $\tau(\lambda) \ge r$ and $\tau^*(\lambda) \le s$, where $r \in I_0$ and $s \in I_1$. Then $bC_{\tau,\tau^*}(\lambda,r,s) \le \lambda$. But, $\mu \le \lambda$ so,

$$bC_{\tau,\tau^*}(\mu,r,s) \le bC_{\tau,\tau^*}(\lambda,r,s) \le \lambda.$$

Also, since μ is an (r, s)-gfbc relative to λ, then

$$\lambda \wedge bC_{\tau,\tau^*(\lambda)}(\mu,r,s) = bC_{\tau,\tau^*}(\mu,r,s),$$

so

$$bC_{\tau,\tau^*}(\mu,r,s) = bC_{\tau,\tau^*(\lambda)}(\mu,r,s) \le \lambda.$$

Now, if μ is an (r, s)-gfbc relative to λ and $\tau(\alpha) \ge r$ and $\tau^*(\alpha) \le s$ where $r \in I_0$ and $s \in I_1$ such that $\mu \le \alpha$, then for each an (r, s)-fo set $\alpha \wedge \lambda$, $\mu = \mu \wedge \lambda \le \alpha \wedge \lambda$. Hence μ is an (r, s)-gfbc relative to λ,

$$bC_{\tau,\tau^*}(\mu,r,s) = bC_{\tau,\tau^*(\lambda)}(\mu,r,s) \le (\alpha \wedge \lambda) \le \alpha.$$

Therefore, μ is an (r, s)-gfbc in I^X.

Conversely, let μ be an (r, s)-gfbc set in I^X and $\tau(\alpha) \ge r$ and $\tau^*(\alpha) \le s$ whenever $\alpha \le \lambda$ such that $\mu \le \alpha$, $r \in I_0$ and $s \in I_1$. Then for each an (r, s)-fo set $\beta \in I^X$, $\alpha = \beta \wedge \lambda$. But we have, μ is an (r, s)-gfbc set in I^X such that $\mu \le \beta$,

$$bC_{\tau,\tau^*}(\mu,r,s) \le \beta \Rightarrow bC_{\tau,\tau^*(\lambda)}(\mu,r,s) = bC_{\tau,\tau^*}(\mu,r,s) \wedge \lambda \le \beta \wedge \lambda = \alpha.$$

That is, μ is an (r, s)-gfbc relative to λ.

(5) Suppose μ is an (r, s)-fbo and $\lambda \bar{q} \mu$, $r \in I_0$ and $s \in I_1$. Then $\lambda \le (\underline{1}-\mu)$. Since $(\underline{1}-\mu)$ is an (r, s)-fbc set of I^X and λ is an (r, s)-gfbc set, then

$$bC_{\tau,\tau^*}(\lambda,r,s) \bar{q} \mu.$$

Conversely, let μ be an (r, s)-fbc set of I^X such that $\lambda \le \mu$, $r \in I_0$ and $s \in I_1$. Then

$$\lambda \bar{q} (\underline{1}-\mu).$$

But

$$bC_{\tau,\tau^*}(\lambda,r,s) \bar{q} (\underline{1}-\mu) \Rightarrow bC_{\tau,\tau^*}(\lambda,r,s) \le \mu.$$

Hence λ is an (r, s)-gfbc.

Proposition 3.1. Let (X, τ, τ^*) be a dfts, $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$.

1. If λ is an (r, s)-gfbc and an (r, s)-fbo set, then λ is an (r, s)-fbc set.
2. If λ is an (r, s)-fo and an (r, s)-gfbc, then $\lambda \wedge \mu$ is an (r, s)-gfbc set whenever $\mu \le bC_{\tau,\tau^*}(\lambda,r,s)$.

Proof. (1) Suppose λ is an (r, s)-gfbc and an (r, s)-fbo set such that $\lambda \le \lambda$, $r \in I_0$ and $s \in I_1$. Then

$$bC_{\tau,\tau^*}(\lambda,r,s) \le \lambda.$$

But we have,

$$\lambda \le bC_{\tau,\tau^*}(\lambda,r,s).$$

Then,

$$\lambda = bC_{\tau,\tau^*}(\lambda,r,s).$$

Therefore, λ is an (r, s)-fbc set.

(2) Suppose that λ is an (r, s)-fo and an (r, s)-gfbc set, $r \in I_0$ and $s \in I_1$. Then

$$bC_{\tau,\tau^*}(\lambda,r,s) \le \lambda \Rightarrow \lambda \text{ is an } (r,s)\text{-fbc set}$$
$$\Rightarrow \lambda \wedge \mu \text{ is an } (r,s)\text{-fbc}$$
$$\Rightarrow \lambda \wedge \mu \text{ is an } (r,s)\text{-gfbc.}$$

4. (r, s)-generalized ⋆-fuzzy *b*-closed sets

In this section, we introduce and study some properties of a new class of fuzzy sets called an (r, s)-generalized ⋆-fuzzy closed sets and an (r, s)-generalized ⋆-fuzzy *b*-closed sets

Definition 4.1. Let (X, τ, τ^*) be a dfts. For each $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$. A fuzzy set λ is called:

1. An (r, s)-generalized $\star$-fuzzy closed (briefly, (r, s)-g *fc) if $C_{\tau,\tau^*}(\lambda, r, s) \le \mu$ whenever $\lambda \le \mu$ and μ is an (r, s)-gfo set in I^X. λ is called an (r, s)-generalized $\star$-fuzzy open (briefly, (r, s)-g *fo) iff $\underline{1} - \lambda$ is (r, s)-g *fc set.
2. An (r, s)-generalized $\star$-fuzzy b-closed (briefly, (r, s)-g *fbc) if $bC_{\tau,\tau^*}(\lambda, r, s) \le \mu$ whenever $\lambda \le \mu$ and μ is an (r, s)-gfo set in I^X. λ is called an (r, s)-generalized $\star$-fuzzy b-open (briefly, (r, s)-g *fbo) iff $\underline{1} - \lambda$ is (r, s)-g *fbc set.

Theorem 4.1. Let (X, τ, τ^*) be a dfts. $\lambda \in I^X$ is an (r, s)-g *fbo set if and only if $\mu \le bI_{\tau,\tau^*}(\lambda, r, s)$ whenever μ is an (r, s)-gfc, $r \in I_0$ and $s \in I_1$.

Proof. Suppose that λ is an (r, s)-g $\star$fbo set in I^X, and let μ is an (r, s)-gfc set such that $\mu \le \lambda$, $r \in I_0$ and $s \in I_1$. So by the definition, we have $\underline{1} - \lambda$ is an (r, s)-gfo set in I^X and $\underline{1} - \lambda \le \underline{1} - \mu$. But $\underline{1} - \lambda$ is an (r, s)-g $\star$fbc set, then $bC_{\tau,\tau^*}(\underline{1} - \lambda, r, s) \le \underline{1} - \mu$. But

$$bC_{\tau,\tau^*}(\underline{1} - \lambda, r, s) = \underline{1} - bI_{\tau,\tau^*}(\lambda, r, s) \le \underline{1} - \mu.$$

Therefore,

$$\mu \le bI_{\tau,\tau^*}(\lambda, r, s).$$

Conversely, suppose that $\mu \le bI_{\tau,\tau^*}(\lambda, r, s)$ whenever $\mu \le \lambda$ and μ is an (r, s)-gfc set, $r \in I_0$ and $s \in I_1$. Now

$$\underline{1} - bI_{\tau,\tau^*}(\lambda, r, s) \le \underline{1} - \mu,$$

Thus

$$bC_{\tau,\tau^*}(\underline{1} - \lambda, r, s) \le \underline{1} - \mu.$$

Therefore, $\underline{1} - \lambda$ is an (r, s)-gfbc set and λ is an (r, s)-gfbo set.

Proposition 4.1. Let (X, τ, τ^*) be dfts's. For each $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$

1. If a fuzzy set λ is an (r, s)-g *fbc, then $bC_{\tau,\tau^*}(\lambda, r, s) - \lambda$ contains no non-zero (r, s)-gfc set.
2. If a fuzzy set λ is an (r, s)-g *fbc, then $bC_{\tau,\tau^*}(\lambda, r, s) - \lambda$ is an (r, s)-g $\star$fbo.
3. An (r, s)-g $\star$fbc set λ is an (r, s)-fbc iff $bC_{\tau,\tau^*}(\lambda, r, s) - \lambda$ is an (r, s)-fbc set.
4. If a fuzzy set λ is an (r, s)-g $\star$fbc, then $\mu = \underline{1}$, whenever μ is an (r, s)-gfo set and $bI_{\tau,\tau^*}(\lambda, r, s) \vee (\underline{1} - \lambda) \le \mu$.

Proof. (1) Suppose that λ is an (r, s)-g $\star$fbc set and μ is an (r, s)-gfc set of I^X, $r \in I_0$ and $s \in I_1$ such that

$$\mu \le bC_{\tau,\tau^*}(\lambda, r, s)$$

And

$$\lambda \le \underline{1} - \mu.$$

But λ is an (r, s)-g $\star$fbc set and $\underline{1} - \mu$ is an (r, s)-gfo set, then

$$bC_{\tau,\tau^*}(\lambda, r, s) \le \underline{1} - \mu \Rightarrow \mu \le bC_{\tau,\tau^*}(\lambda, r, s) \wedge (\underline{1} - bC_{\tau,\tau^*}(\lambda, r, s)) = \underline{0}.$$

Therefore $bC_{\tau,\tau^*}(\lambda, r, s) - \lambda$ contains no non-zero (r, s)-gfc set.

(2) Let λ be an (r, s)-g $\star$fbc set, $r \in I_0$ and $s \in I_1$. Then by (1) we have, $bC_{\tau,\tau^*}(\lambda, r, s) - \lambda$ contains no non-zero (r, s)-gfc set. So, $bC_{\tau,\tau^*}(\lambda, r, s) - \lambda$ is an (r, s)-g $\star$fbo set.

(3) Let λ be an (r, s)-g $\star$fbc set. If λ is an (r, s)-fbc, $r \in I_0$ and $s \in I_1$, then

$$bC_{\tau,\tau^*}(\lambda, r, s) - \lambda = \underline{0}.$$

Conversely, let $bC_{\tau,\tau^*}(\lambda, r, s) - \lambda$ is an (r, s)-fbc set in I^X and λ is an (r, s)-g $\star$fbc, $r \in I_0$ and $s \in I_1$, then by (1) we have, $bC_{\tau,\tau^*}(\lambda, r, s) - \lambda$ contains no non-zero (r, s)-gfc set. Then,

$$bC_{\tau,\tau^*}(\lambda, r, s) - \lambda = \underline{0},$$

that is

$$bC_{\tau,\tau^*}(\lambda, r, s) = \lambda.$$

Hence λ is an (r, s)-fbc set.

(4) Let μ be an (r, s)-gfc set and $bI_{\tau,\tau^*}(\lambda, r, s) \vee (\underline{1} - \lambda) \le \mu$, $r \in I_0$ and $s \in I_1$. Hence

$$\underline{1} - \mu \le bC_{\tau,\tau^*}(\underline{1} - \lambda, r, s) \wedge \lambda = bC_{\tau,\tau^*}(\underline{1} - \lambda, r, s) - (\underline{1} - \lambda).$$

But $(\underline{1} - \mu)$ is an (r, s)-gfc and $\underline{1} - \lambda$ is an (r, s)-g $\star$fbc by (1), $\underline{1} - \mu = \underline{0}$ and hence $\mu = \underline{1}$.

Proposition 4.2. Let (X, τ, τ^*) be dfts's. For each λ and $\mu \in I^X$, $r \in I_0$ and $s \in I_1$.

If λ and μ are (r, s)-g $\star$fbc, then $\lambda \wedge \mu$ is an (r, s)-g $\star$fbc. 1.

2. If λ is an (r, s)-g $\star$fbc and $\tau(\mu) \ge r$, $\tau^*(\mu) \le s$, then $\lambda \wedge \mu$ is an (r, s)-g $\star$fbc.

Proof. (1) Suppose that λ and μ are (r, s)-g $\star$fbc sets in I^X such that $\lambda \wedge \mu \le \nu$ for each an (r, s)-gfo set $\nu \in I^X$, $r \in I_0$ and $s \in I_1$. Since λ is an (r, s)-g $\star$fbc,

$$bC_{\tau,\tau^*}(\lambda, r, s) \le \nu$$

for each an (r, s)-gfo set $\nu \in I^X$ and $\lambda \le \nu$. Also, μ is an (r, s)-g $\star$fbc,

$$bC_{\tau,\tau^*}(\mu, r, s) \le \nu$$

for each an (r, s)-gfo set $\nu \in I^X$ and $\mu \le \nu$. Then we have,

$$bC_{\tau,\tau^*}(\lambda, r, s) \wedge bC_{\tau,\tau^*}(\mu, r, s) \le \nu,$$

whenever $\lambda \wedge \mu \le \nu$, Therefore, $\lambda \wedge \mu$ is an (r, s)-g $\star$fbc.

(2) Since every an (r, s)-fc set is an (r, s)-g $\star$fbc and from (1) we get the proof.

Proposition 4.3. Let (X, τ, τ^*) be dfts's. For each λ and $\mu \in I^X$, $r \in I_0$ and $s \in I_1$.

1. If λ is both an (r, s)-gfo and an (r, s)-g $\star$fbc, then λ is an (r, s)-fbc set.
2. If λ is an (r, s)-g $\star$fbc and $\lambda \le \mu \le bC_{\tau,\tau^*}(\lambda, r, s)$, then μ is an (r, s)-g $\star$fbc.

Proof. (1) Suppose that λ is an (r, s)-gfo and an (r, s)-g $\star$fbc in I^X such that $bC_{\tau,\tau^*}(\lambda, r, s) \le \mu$, $r \in I_0$ and $s \in I_1$. But

$$\lambda \le bC_{\tau,\tau^*}(\lambda, r, s).$$

Therefore

$$\lambda = bC_{\tau,\tau^*}(\lambda, r, s).$$

Hence λ is an (r, s)-fbc set.

(2) Suppose that λ is an (r, s)-g $\star$fbc and ν is an (r, s)-gfo set in I^X such that $\mu \le \nu$ for each $\mu \in I^X$, $r \in I_0$ and $s \in I_1$. So $\lambda \le \nu$. But we have, λ is an (r, s)-g $\star$fbc, then

$$bC_{\tau,\tau^*}(\lambda, r, s) \le \nu.$$

Now

$$bC_{\tau,\tau^*}(\mu, r, s) \le bC_{\tau,\tau^*}(bC_{\tau,\tau^*}(\lambda, r, s), r, s) = bC_{\tau,\tau^*}(\lambda, r, s) \le \nu.$$

Therefore μ is an (r, s)-g $\star$fbc set.

Theorem 4.2. Let (X, τ_1, τ_1^*) and (Y, τ_2, τ_2^*) be dfts's. If $\lambda \le \underline{1}_Y \le \underline{1}_X$ such that λ is an (r, s)-g $\star$fbc in I^X, $r \in I_0$ and $s \in I_1$, then λ is an (r, s)-g $\star$fbc relative to Y.

Proof. Suppose that (X, τ_1, τ_1^*) and (Y, τ_2, τ_2^*) are dfts's such that $\lambda \le \underline{1}_Y \le \underline{1}_X$, $r \in I_0$, $s \in I_1$ and λ is an (r, s)-g $\star$fbc in I^X. Now, let $\lambda \le \underline{1}_Y \wedge \mu$ such that μ is an (r, s)-gfo set in I^X. But we have, λ is an (r, s)-g *fbc in I^X,

$$\lambda \le \mu \Rightarrow bC_{\tau,\tau^*}(\lambda, r, s) \le \mu.$$

So that

$$\underline{1}_Y \wedge bC_{\tau,\tau^*}(\lambda, r, s) \le \underline{1}_Y \wedge \mu.$$

Hence λ is an (r, s)-g $\star$fbc relative to Y.

Theorem 4.3. Let (X, τ_1, τ_1^*) be adfts. For each λ and $\mu \in I^X$, $r \in I_0$ and $s \in I_1$ with $\mu \le \lambda$. If μ is an (r, s)-g $\star$fbc relative to λ such that λ is both an (r, s)-gfo and (r, s)-g $\star$fbc of I^X, then μ is an (r, s)-g $\star$fbc relative to X.

Proof. Suppose that μ is an (r, s)-g $\star$fbc and $\tau(\nu) \ge r$ and $\tau^*(\nu) \le s$ such that $\mu \le \nu$, $r \in I_0$, $s \in I_1$. But we have, $\mu \le \lambda \le \underline{1}$, therefore $\mu \le \lambda$ and $\mu \le \nu$. So

$$\mu \le \lambda \wedge \nu.$$

Also we have, μ is an (r, s)-g $\star$fbc relative to λ,

$$\lambda \wedge bC_{\tau,\tau^*}(\mu, r, s) \le \lambda \wedge \nu \Rightarrow \lambda \wedge bC_{\tau,\tau^*}(\mu, r, s) \le \nu.$$

Thus

$$(\lambda \wedge bC_{\tau,\tau^*}(\mu, r, s)) \vee (\underline{1} - bC_{\tau,\tau^*}(\mu, r, s)) \le \nu \vee (\underline{1} - bC_{\tau,\tau^*}(\mu, r, s)).$$
$$\Rightarrow \lambda \vee (\underline{1} - bC_{\tau,\tau^*}(\mu, r, s)) \le \nu \vee (\underline{1} - bC_{\tau,\tau^*}(\mu, r, s)).$$

Since λ is an (r, s)-g $\star$fbc, then

$$bC_{\tau,\tau^*}(\lambda, r, s) \le \nu \vee (\underline{1} - \mu).$$

Also,

$$\mu \le \lambda \Rightarrow bC_{\tau,\tau^*}(\mu, r, s) \le bC_{\tau,\tau^*}(\lambda, r, s).$$

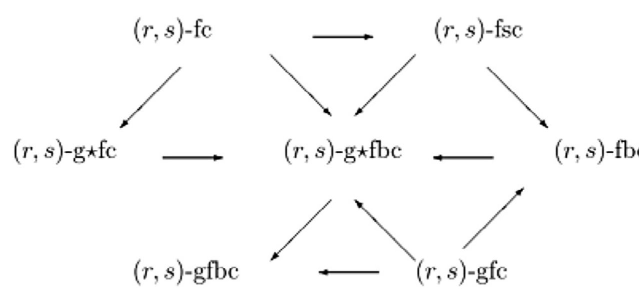
Thus

$$bC_{\tau,\tau^*}(\mu, r, s) \le bC_{\tau,\tau^*}(\lambda, r, s) \le \nu \vee (\underline{1} - bC_{\tau,\tau^*}(\mu, r, s)).$$

Therefore $bC_{\tau,\tau^*}(\mu, r, s) \le \nu$, but $bC_{\tau,\tau^*}(\mu, r, s)$ is not contained in $(\underline{1} - bC_{\tau,\tau^*}(\mu, r, s))$. That is, μ is an (r, s)-g $\star$fbc relative to X.

5. Interrelations

The following implication illustrates the relationships between different fuzzy sets:



None of these implications is reversible where A → B represents A implies B, as shown by the following examples. But at this stage we do not have information regarding the relationship between an (r, s)-gfbc and (r, s)-g $\star$fc sets.

Example 5.1. (1) Let $X = \{a, b, c\}$ and let μ and α are fuzzy sets defined by:

$$\mu(a) = 1.0, \quad \mu(b) = 0.5, \quad \mu(c) = 0.0,$$

$$\alpha(a) = 0.0, \quad \alpha(b) = 0.4, \quad \alpha(c) = 1.0.$$

Define (τ, τ^*) on X as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases} \qquad \tau^*(\lambda) = \begin{cases} 0, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ 1, & \text{otherwise.} \end{cases}$$

Then α is an $(\frac{1}{2}, \frac{1}{2})$-gfbc set, but not an $(\frac{1}{2}, \frac{1}{2})$-g $\star$fbc set.

(2) Take $X = \{a, b\}$ in (1) and define μ, α and β by:

$$\mu(a) = 0.6, \quad \mu(b) = 0.6,$$

$$\alpha(a) = 0.3, \quad \alpha(b) = 0.2,$$

$$\beta(a) = 0.4, \quad \beta(b) = 0.5.$$

Then β is an $(\frac{1}{2}, \frac{1}{2})$-g $\star$fbc set, but not an $(\frac{1}{2}, \frac{1}{2})$-fbc set.

(3) Let $X = \{a, b, c\}$. Define μ, ν and γ by:

$$\mu(a) = 1.0, \quad \mu(b) = 0.5, \quad \mu(c) = 0.3,$$

$$\nu(a) = 1.0, \quad \nu(b) = 0.6, \quad \nu(c) = 0.0,$$

$$\gamma(a) = 0.0, \quad \gamma(b) = 0.6, \quad \gamma(c) = 0.0.$$

Define (τ, τ^*) as in (1). Then ν is an $(\frac{1}{2}, \frac{1}{2})$-g $\star$fbc set but not an $(\frac{1}{2}, \frac{1}{2})$-fc set and not an $(\frac{1}{2}, \frac{1}{2})$-gfc. And γ is an $(\frac{1}{2}, \frac{1}{2})$-g $\star$fbc set, but not an $(\frac{1}{2}, \frac{1}{2})$-fsc set.

(4) Take (3) and defined μ and ν by:

$$\mu(a) = 1.0, \quad \mu(b) = 1.0, \quad \mu(c) = 0.6,$$

$$\nu(a) = 0.3, \quad \nu(b) = 0.5, \quad \nu(c) = 0.5.$$

Define (τ, τ^*) as in (1). Then ν is an $(\frac{1}{2}, \frac{1}{2})$-g $\star$fbc set, but not an $(\frac{1}{2}, \frac{1}{2})$-g $\star$fc set.

(5) See Example 3.1. Clearly β is an $(\frac{1}{2}, \frac{1}{2})$-gfbc set, but not an $(\frac{1}{2}, \frac{1}{2})$-gfc set.

(6) Let $X = \{a, b\}$. Define μ, ν and γ as follows:

$$\mu(a) = 0.7, \quad \mu(b) = 0.6,$$

$$\nu(a) = 0.3, \quad \nu(b) = 0.2,$$

$$\gamma(a) = 0.4, \quad \gamma(b) = 0.5.$$

Define (τ, τ^*) as in (1). Then ν is an $(\frac{1}{2}, \frac{1}{2})$-fbc set but not an $(\frac{1}{2}, \frac{1}{2})$-fsc set, also not an $(\frac{1}{2}, \frac{1}{2})$-gfc.

(7) Let $X = \{a, b, c\}$ and let μ and α as fuzzy sets defined by:

$$\mu(a) = 0.9, \quad \mu(b) = 0.8, \quad \mu(c) = 0.3,$$

$$\alpha(a) = 0.1, \quad \alpha(b) = 0.8, \quad \alpha(c) = 0.3.$$

Define (τ, τ^*) on X by:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ 0.6, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases} \qquad \tau^*(\lambda) = \begin{cases} 0, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ 0.3, & \text{if } \lambda = \mu, \\ 1, & \text{otherwise.} \end{cases}$$

Then α is an $(0.6, 0.3)$-fsc set, but not an $(0.6, 0.3)$-fc set.

(8) Let $X = \{a, b\}$ and let μ and α as fuzzy sets defined by:

$$\mu(a) = 0.9, \quad \mu(b) = 0.4,$$

$$\alpha(a) = 0.1, \quad \alpha(b) = 0.8.$$

Define (τ, τ^*) on X by:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases} \qquad \tau^*(\lambda) = \begin{cases} 0, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ 1, & \text{otherwise.} \end{cases}$$

Then μ is an $(\frac{1}{2}, \frac{1}{2})$-g $\star$fc set, but not an $(\frac{1}{2}, \frac{1}{2})$-fc set.

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REFERENCES

[1] Zadeh LA. Fuzzy sets. Inf Control 1965;8:338–53.
[2] Atanassov K. New operators defined over the intuitionistic fuzzy sets. Fuzzy Set Syst 1993;61:131–42.
[3] Çoker D. An introduction to intuitionistic fuzzy topological spaces. Fuzzy Set Syst 1997;88:81–9.
[4] Samanta SK, Mondal TK. On intuitionistic gradation of openness. Fuzzy Set Syst 2002;131:323–36.
[5] Gutiérrez Garcia J, Rodabaugh SE. Order-theoretic, topological, categorical redundancies of interval-valued sets, grey sets, vague sets, interval-valued; intuitionistic sets, intuitionistic fuzzy sets and topologies. Fuzzy Set Syst 2005;156:445–84.
[6] Omari AA, Noorani MSM. On generalized b-closed sets. Bull Malays Math Sci Soc 2009;32:19–30.
[7] Mohammed FM, Noorani MSM, Ghareeb A. Several types of totally continuous functions in double fuzzy topological spaces. Adv Fuzzy Syst 2014;2014:Article ID 361398, 9 pages.
[8] Lee EP, Im YB. Mated fuzzy topological spaces. J Fuzzy Log Int Sys 2001;11:161–5.
[9] Kim YC, Abbas SE. Several types of fuzzy regular spaces. Indian J Pure Appl Math 2004;35:481–500.
[10] Abbas SE, El-Sanousy E. Several types of double fuzzy semiclosed sets. J Fuzzy Math 2012;20:89–102.
[11] Mohammed FM, Noorani MSM, Ghareeb A. Generalized $\psi\rho$-operations on double fuzzy topological spaces. AIP Conf Proc 2013;1571:943–8.
[12] Mohammed FM, Noorani MSM, Ghareeb A. Generalized $\psi\rho$-closed sets and generalized $\psi\rho$-open sets in double fuzzy topological spaces. AIP Conf Proc 2014;1602:909–17.